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A GENERAL THEOREM IN LOCAL PROBABILITY.

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The theorem which forms the objective point of this paper deals with the probability that of n points, distributed at random on a line of length a, the distance between every two shall exceed an arbitrarily assumed distance b. The proof was suggested by the following problem given by Czuber.*

Problem: *In a straight line $AB=a$ two points are assumed at random. It is reqnired to find the probability that their distance apart shall exceed a given length b.*

Czuber gives two solutions which are essentially as follows:

First Solution. The probability that the first point lies within an interval δx_1 is $\delta x_1/a$. The probability that the second point lies within an interval δx_2 is $\delta x_2/a$. The probability that the two points lie simultaneously within these respective intervals is $\delta x_1 \delta x_2/a^2$.

Let each interval be diminished indefinitely, the limiting value $\delta x_1 \delta x_2/a^2$ represents the probability that two points, placed at random on the line AB, are situated at the respective distances x_1 and x_2 from A.

All positions on the line are mutually exclusive and equally possible. The positions favorable to the conditions of our problem are subject to the restriction that the difference between x_1 and $x_2 > b$.

Suppose $x_1 < x_2$, and let x_1 be taken at random. Then every value of x_2 between the limits x_1+b and a gives a favorable case, while x_1 may have any value from 0 to $a-b$. The probability of the favorable cases subject to the condition $x_1 < x_2$ is therefore given by the double integral

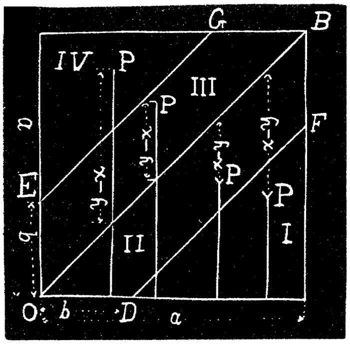
$$\int_0^{a-b}\int_{x_1+b}^{a}\frac{dx_1dx_2}{a_2}=\frac{1}{a^2}\int_0^{a-b}(a-b-x_1)dx_1=\tfrac{1}{2}\left(\frac{a-b}{a}\right)^2$$

**Geometrische Wahrscheinlichkeiten*, u. Mittelwerte, §11.

and remembering that for every favorable case $x_1 < x_2$ there is another for which $x_1 > x_2$, we have for the required probability

$$P=\left(\frac{a-b}{a}\right)^2.$$

The second solution is geometrical. As before denote by x_1 and x_2 the distances of two points from A., Next construct a square having the length a for a side, and let P be any point within this square whose coördinates, with reference to a pair of intersecting sides of the square for axes, are x and y.



There is now a one-to-one correspondence between the coördinates x, y and the distances x_1, x_2 of the points on the line from A, to every point in the square there will correspond one and only one point pair on the line and vice versa. The differences between the coördinates x, y correspond to the distances between the prsitions on the line. We can therefore find the required probability by comparing the area of the region of the square for which the difference between x and $y > b$ with the area of the whole square. Take $OD=OE=b$, and draw DF and EG each parallel to the diagonal OB. Denote the parts into which the square is divided by these lines by I, II, III, IV, respectively. Then plainly for every point in

$$\begin{array}{rl} \text{I} & x-y>b, \\ \text{II} & x-y<b, \\ \text{III} & y-x<b, \\ \text{IV} & y-x>b, \end{array}$$

that is, the strips I and IV contain all the points satisfying thr conditions of our problem and no other points.

Now I and II combined constitute a square whose side is $a-b$, and consequently we have for the required probability

$$P=\left(\frac{a-b}{a}\right)^2.$$

Each of these proofs can be readily extended to the case of three points. We will state this case as

Problem 2. *In a straight line $AB=a$ three points are assumed at random. What is the probability that the distance between each pair of points exceeds a given distance b?*

First Solution. Let P_1, P_2, P_3 represent the three points and x_1, x_2, x_3 their respective distances from the point A.

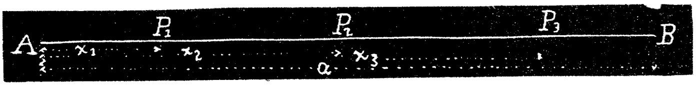
With reference to the positions of the points relative to each other six mutually exclusive cases are equally possible, namely,

$$x_1 < x_2 < x_3,$$
$$x_1 < x_3 < x_2,$$
$$x_2 < x_3 < x_1,$$
$$x_2 < x_1 < x_3,$$
$$x_3 < x_1 < x_2,$$
$$x_3 < x_2 < x_1,$$

consequently the total number of favorable cases is equal to six times the number of favorable cases—subject to one of these conditions, say to $x_1 < x_2 < x_3$.

Reasoning precisely as in the case of two points we see that the possibility that P_1, P_2, P_3 are at the distances x_1, x_2, x_3 from A is $dx_1 dx_2 dx_3/a^3$, and that the required probability is the triple integral of this expression, the limits of integration being so chosen as to include all favorable and to exclude all unfavorable positions. For $x_1 < x_2 < x_3$ these limits are

for P_3 from $x_2 + b$ to a,
for P_2 from $x_1 + b$ to $a - b$,
for P_1 from 0 to $a - 2b$.



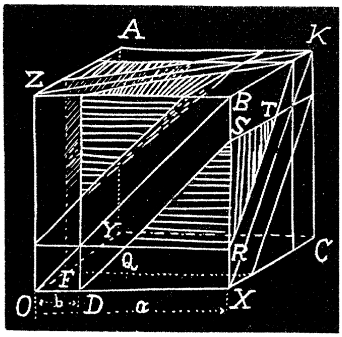
$$\therefore P = \frac{6}{a^3}\int_0^{a-2b}\int_{x_1+b}^{a-b}\int_{x_2+b}^{a} dx_1\,dx_2\,dx_3 = \int_0^{a-2b}\int_{x_1+b}^{a-b}(a-b-x_2)\,dx_1\,dx_2$$
$$= \frac{3}{a^3}\int_0^{a-2b}(a-2b-x_1)^2\,dx_1 = \left(\frac{a-2b}{a}\right)^3.$$

Second Solution. Consider a cube of side a. Take one set of concurrent edges of this cube for axes of coördinates and denote by x, y, z the coördinates of any point P within this cube. To every triplet of points P_1, P_2, P_3 on the line AB corresponds a single point in the cube, viz, the point whose coördinates are $x = x_1$, $y = x_2$, $z = x_3$; to every point in the cube corresponds a single triplet of points on the line, viz, the points whose distances from A are respectively $x_1 = x$, $x_2 = y$, $x_3 = z$. The one-to-one correspondence between the points in the cube and the different portions of three points on the line being established, the problem is solved by comparing the volume of the regions of the cube for which the coördinates of the points satisfy the simultaneous conditions

the difference between x and $y > b$,
the difference between y and $z > b$,
the difference between z and $x > b$,

with the volume of the entire cube.

Take $OD=OE=OF=b$, and through each of the points F and D draw planes parallel to the diagonal plane $OZKC$.



For every point between the planes thus drawn we have the difference between x and $y<b$; for every point in these planes the difference between x and $y=b$; and for the points exterior to the planes we have the difference between x and $y>b$. Again, if planes be drawn through each of the points E and F parallel to the diagonal plane $OXKA$, we see that the difference between y and $z>b$ only for points which lie exterior to both of these planes, and finally in order that the difference between z and $x>b$ the points must be exterior to the planes drawn through E and D respectively, parallel to $OYKB$. The points which satisfy simultaneously the three conditions

the difference between x and $y>b$,
the difference between y and $z>b$,
the difference between z and $x>b$,

are therefore limited to those regions of the cube which lie exterior to each of these three sets of parallel planes, that is to six equal tetrahedrons such as $QRST$.

The dimensions of this tetrahedron are $QR=RS=ST=a-2b$, since $BS=OD=a$ and also $RX=OE=a$, and the volume of $QRST=\frac{1}{6}(a-2b)^3$.

The required probability is therefore

$$P=\frac{6QRST}{\text{volume of entire cube}}=\left(\frac{a-2b}{a}\right)^3.$$

We will now prove the general

Theorem. *The probability that of n points, distributed at random along a line of length a, no two shall fall within the distance b of each other is*

$$\left[\frac{a-(n-1)b}{a}\right]^n.$$

We denote by x_1, x_2, x_3, ..., x_n the distances of the n points from one extremity of the line a, and first compute the probability that the distance between every pair of points exceeds b when the points are subject to the condition

$$x_1<x_2<x_3<\dots\dots<x_n.$$

Denote this probability by Q. Then since the suffixes 1, 2, 3, ..., n, may be permuted in $n!$ number of ways, each permutation giving rise to a different distribution of points equally likely, we shall have for the required probability $P=n!Q$. Reasoning exactly as in the case of three points we see that

$$Q=\int_0^{a-(n-1)b}\int_{x_1+b}^{a-(n-2)b}\cdots\cdots\int_{x_{n-2}+b}^{a+b}\int_{x_{n-1}+b}^{a}\frac{dx_1}{a}\frac{dx_2}{a}\cdots\cdots\frac{dx_{n-1}}{a}\frac{dx_n}{a},$$

or if we drop suffixes, since there is no danger of confusion,

$$Q=\int_0^{a-(n-1)b}\int_{x+b}^{a-(n-1)b}\cdots\cdots\int_{x+b}^{a-b}\int_{x+b}^{a}\frac{dx^n}{a^r}.$$

To evaluate this integral we observe that in the evaluation of

$$\int_{x+b}^{a-2b}\int_{x+b}^{a-b}\int_{x+b}^{a}dx^3=\int_{x+b}^{a-2b}\int_{x+b}^{a-b}(a-b-x)dx^2=\int_{x+b}^{a-2b}\tfrac{1}{2}(a-2b-x)^2dx$$

$$=\frac{1}{3!}(a-3b-x)^3,$$

occur the expressions $a-b-x$, $\frac{1}{2}(a-2b-x)^2$, $\frac{1}{3!}(a-3b-x)^3$

Let $u_k=\frac{1}{k!}(a-kb-x)^k$, $u_0=1$. Then

$$\int_{x+b}^{a-kb}u_kdx=\int_{x+b}^{a-kb}\frac{1}{k!}(a-kb-x)^kdx=\frac{1}{(k+1)!}[a-(k+1)b-x]^{k+1}=u_{k+1},$$

and hence we have the chain of expressions,

$$u_{n-1}=\int_{x+b}^{a-(n-2)b}u_{n-2}dx,\quad u_{n-2}=\int_{x+b}^{a-(n-3)b}u_{n-3}dx,\ \cdots\cdots,\ u_1=\int_{x+b}^{a}u_0dx=\int_{x+b}^{a}dx.$$

Eliminating successively the quantities u_{n-2}, u_{n-3}, ..., u_1 on the right, and substituting for u_{n-1} its value, we obtain

$$u_{n-1}=\int_{x+b}^{a-(n-2)b}\cdots\cdots\int_{x+b}^{a-b}\int_{x+b}^{a}dx^n=\frac{1}{(n-1)!}[a-(n-1)b-x]^{n-1},$$

$$Q=\frac{1}{a^n}\int_0^{a-(n-1)b}u_{n-1}dx=\frac{1}{a^n}\int_0^{a-(n-1)b}[a-(n-1)b-x]^{n-1}dx$$

$$=\frac{1}{a^n}\cdot\frac{1}{n!}[a-(n-1)b]^n.$$

$$\therefore P=n!Q=\left(\frac{a-(n-1)b}{a}\right)^n.$$

Corollary I. The probability that of n points, distributed at random on a line of length a, at least two shall fall within a distance b of each other is $1-P$.

Corollary II. The probability p that of n points, distributed at random on a closed curve of length a, no two fall within an arc b of each other is

$$p=\left(\frac{a-nb}{a}\right)^{n-1}.$$

For let one point be located at random and let the arc be cut at that point and straightened. No point which falls within a distance b of either end of this line gives a favorable case. We may therefore consider the favorable cases in the distribution of the remaining $n-1$ points on the line of length $a-2b$. The probability therefore involves one less integral sign and the limit of variation is not a but $a-2b$. The denominator however remains a, since this is the measure of the total number of possible cases. Replacing therefore n by $n-1$ in the expression P, and the a in the numerator by $a-2b$ we obtain p.

Corollary III. The probability that of n points distributed at random on a closed curve of length a, at least two fall within a distance b of each other is $1-p$.

NUMERALS FOR SIMPLIFYING ADDITION.

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Primitive numerals are composed of the sign for unity repeated as many times as there are units in the numbers represented. Systems in which all nine figures are primitive were used by the Egyptians in their hieroglyphic writing, the Phœnicians, the Babylonians in their cuneiform inscriptions, the early Romans, and probably the early Indians. By virtue of such a notation, the proeess of addition can be reduced to counting by ones. The meaning of the symbols being self-evident does not have to be learned.*

In many instances, the first two, three, or four numerals are essentially primitive while the others are more arbitrary or superficial; *e. g.*, the hieratic symbols of the Egyptians, the ordinary Roman numrals, the moderately early Indian numerals, and the numerals now used by the Chinese.

The Palmyrenes made use of two component signs in constructing the first nine numerals,—simple strokes representing the ones and a certain Y-like symbol the component five found in each of the numbers five, six, seven, eight, and nine. Here addition can be performed by counting by ones and by fives,—the fives, for convenience, being taken in pairs so that most of the counting is really by tens instead of by fives. The same is true of the Roman numerals of the republican period where IIII is written for IV and VIIII for IX.

*Columns 3. 5. 6. and 7 of the accompanying figure are copied from the Encyclopedia Britannica, article "Numerals."